

# Entropy of Reissner-Nordstrom Black Holes with Minimal Length Revisited

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(Dated: February 1, 2008)

## Abstract

Based on the generalized uncertainty principle, we study the entropy of a four-dimensional black hole by counting degrees of freedom near the horizon and obtain the (finite) entropy proportional to the surface area at the horizon without a cutoff introduced in the conventional brick-wall method.

PACS numbers: 04.70.Dy, 97.60.Lf

Keywords: Generalized Uncertainty Principle; Black Hole; Entropy

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Bekenstein suggested that the entropy of a black hole is proportional to the surface area at the event horizon [1], and subsequently Hawking has proved that based on the quantum field theory the entropy of the Schwarzschild black hole satisfies the area law [2]. Motivated by these works, there have been many interesting works to study the statistical origin of the entropy [3, 4, 5, 6]. One of them is the well-known brick-wall method [7], in which the divergence near the horizon can be removed by introducing a brick-wall cutoff. It can help us to understand the origin of entropy in various black holes [8, 9, 10, 11]. However, the introduction to the brick wall cutoff seems to be more or less unnatural. Recent calculations of the free energy and the entropy have been studied using the generalized uncertainty relation, in which there exists the minimal length [12, 13, 14, 15, 16]. In this regime, the bound of the entropy has been obtained in Ref. [12, 13, 14] and some detail calculations for the specific models are done in Ref. [15, 16]. The simplest way to generalize the uncertainty relation is to promote it to  $\Delta x \Delta p \geq \hbar + \frac{\lambda}{\hbar}(\Delta p)^2$ , which shows that there exists a minimal length,  $\Delta x \geq 2\sqrt{\lambda}$  [17, 18].

Specifically, as for a four-dimensional spherically symmetric black hole [15, 16], the detail higher-order correction to the entropy near the horizon  $r_0$  can be calculated as

$$S = \frac{r_0^2}{12\lambda} \left[ 1 - \frac{4\pi^2}{3} + 64\zeta(3) - \frac{16\pi^4}{5} - \frac{1024\zeta(5)}{3} + O((4\pi)^3) \right]. \quad (1)$$

Note that the higher-order corrections are not negligible, more worse, it is not convergent so that the result may not be reliable. So, in this paper, we would like to obtain the finite convergent entropy of the Reissner-Nordstrom(RN) black hole near the horizon with the generalized uncertainty principle(GUP) in the regime of the brick-wall method. There is no ultraviolet divergence at all as long as we use the generalized uncertainty relation when we count the density of states even at the horizon.

Now, we consider the RN black hole, whose metric is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2, \quad (2)$$

with  $f(r) = 1 - 2M/r + Q^2/r^2$ , where  $M$  and  $Q$  are the mass and the electric charge of the black hole, respectively. Then, the inner ( $r_-$ ) and the outer ( $r_+$ ) horizons are obtained as  $r_{\pm} = M \pm \sqrt{M^2 - Q^2}$ . In this black hole, the Hawking Temperature is  $T = \beta^{-1} = \kappa/(2\pi) = (r_+ - r_-)/(4\pi r_+^2)$ , where  $\kappa$  is the surface gravity at the outer horizon.

The Klein-Gordon equation for a scalar field on this background becomes

$$(\square - \mu^2)\Phi = 0, \quad (3)$$

where  $\mu$  is the mass of the field. Substituting the wave function  $\Phi = \exp(-i\omega t)\psi(r, \theta, \varphi)$  into Eq. (3), we obtain

$$\frac{\partial^2 \psi}{\partial r^2} + \left( \frac{2}{r} + \frac{1}{f} \frac{df}{dr} \right) \frac{\partial \psi}{\partial r} + \frac{1}{f} \left[ \frac{\omega^2}{f} - \mu^2 + \frac{1}{r^2} \left( \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \csc^2 \theta \frac{\partial^2}{\partial \varphi^2} \right) \right] \psi = 0. \quad (4)$$

By using the WKB approximation with  $\psi \sim \exp[iS(r, \theta, \varphi)]$ , we obtain  $p_r^2 = f^{-1} [\omega^2/f - \mu^2 - r^{-2} p_\theta^2 - r^{-2} \csc^2 \theta p_\varphi^2]$ , where  $p_r = \partial S/\partial r$ ,  $p_\theta = \partial S/\partial \theta$ , and  $p_\varphi = \partial S/\partial \varphi$ . And the square module of momentum is given by  $p^2 = p_i p^i = g^{rr} p_r^2 + g^{\theta\theta} p_\theta^2 + g^{\varphi\varphi} p_\varphi^2 = \omega^2/f - \mu^2$ . Then, the number of quantum states is obtained as

$$\begin{aligned} n(\omega) &= \frac{1}{(2\pi)^3} \int dr d\theta d\varphi dp_r dp_\theta dp_\varphi \frac{1}{(1 + \lambda p^2)^3} \\ &= \frac{2}{3\pi} \int dr \frac{r^2 (\omega^2/f - \mu^2)^{3/2}}{\sqrt{f} [1 + \lambda (\omega^2/f - \mu^2)]^3}, \end{aligned} \quad (5)$$

where  $\omega \geq \mu\sqrt{f}$ . Note that it is finite even at the horizon. This is the reason why the brick-wall cutoff is not necessary in our calculation. The free energy for a scalar field is given by  $F = - \int_{\mu\sqrt{f}}^{\infty} d\omega \frac{n(\omega)}{e^{\beta\omega} - 1}$ , and the entropy is obtained as

$$\begin{aligned} S &= \beta^2 \frac{\partial F}{\partial \beta} \\ &= \frac{2\beta^2}{3\pi} \int dr \frac{r^2}{\sqrt{f}} \int_{\mu\sqrt{f}}^{\infty} d\omega \frac{\omega (\omega^2/f - \mu^2)^{3/2}}{4 \sinh^2 \frac{1}{2} \beta \omega [1 + \lambda (\omega^2/f - \mu^2)]^3} \\ &= \frac{\beta^3}{12\pi\lambda^3} \int dr r^2 f \int_{x_0}^{\infty} dx \frac{x (x^2 - x_0^2)^{3/2}}{\sinh^2 x [x^2 - x_0^2 + \beta^2 f / (4\lambda)]^3}, \end{aligned} \quad (6)$$

where  $x \equiv \frac{1}{2}\beta\omega$  and  $x_0 = \frac{1}{2}\beta\mu\sqrt{f}$ .

We now consider a thin layer between  $r_+$  and  $r_+ + \epsilon$  for a nonextremal RN black hole near the horizon for the mode counting since the density of state is dominant in this region. Moreover, since  $x_0$  goes to zero near the horizon, the entropy becomes

$$S = \frac{\beta^3}{12\pi\lambda^3} \int_0^{\infty} dx \frac{x^4}{\sinh^2 x} I(x), \quad (7)$$

where

$$I(x) = \int_{r_+}^{r_+ + \epsilon} dr \frac{r^2 f}{[x^2 + \beta^2 f / (4\lambda)]^3}. \quad (8)$$

Near the horizon, the metric can be simply written as  $f(r) \approx f'(r_+)(r - r_+)$ , where  $f'(r_+) = 2\kappa = (r_+ - r_-)/r_+^2$ . From the metric (2), the minimal length is obtained as

$$2\sqrt{\lambda} \equiv \int_{r_+}^{r_+ + \epsilon} \frac{dr}{\sqrt{f}} = \sqrt{\frac{2\epsilon}{\kappa}}. \quad (9)$$

Then, the integral (8) is calculated as

$$I(x) \approx \frac{r_+^2 f'(r_+) \epsilon^2}{4x^2 [x^2 + \beta^2 f'(r_+) \epsilon / (4\lambda)]^2} = \frac{2\kappa^3 \lambda^2 r_+^2}{x^2 + 4\pi^2}. \quad (10)$$

Substituting Eq. (10) into Eq. (7), we obtain

$$S = \frac{4\pi^2 r_+^2}{3\lambda} \int_0^\infty dx \frac{x^2}{\sinh^2 x (x^2 + 4\pi^2)^2}. \quad (11)$$

Then, the integrand in Eq. (11) can be regarded as a complex function  $h(z) \equiv z^2 / [\sinh^2 z (z^2 + 4\pi^2)^2]$  and poles of  $h(z)$  are given by  $z = in\pi$ , where  $n$ 's are integers except zero. The residues of  $h(z)$  are  $i/(24\pi)$  at  $z = 2i\pi$  and  $-2in(n^2 + 4)/[\pi^3(n^2 - 4)^3]$  at  $z = in\pi$  for  $n \neq 2$ . By the residue theorem, the entropy becomes

$$S = \frac{4\pi^2 r_+^2}{3\lambda} \left( -\frac{1}{24} + \frac{2}{\pi^2} \sum_{\substack{n=1 \\ n \neq 2}}^\infty \frac{n(n^2 + 4)}{(n^2 - 4)^3} \right) = \frac{4\pi r_+^2}{3\lambda} \left( -\frac{\pi}{24} - \frac{25}{32\pi} + \frac{\zeta(3)}{\pi} \right), \quad (12)$$

where this finite result is exact in the near horizon limit. This entropy can be written as  $S = \frac{1}{4} A \cdot \frac{\delta}{\lambda}$ , where  $A = 4\pi r_+^2$  and  $\delta \equiv \frac{1}{3} [\frac{4}{\pi} \zeta(3) - \frac{25}{8\pi} - \frac{\pi}{6}]$ . In order to satisfy the area law of entropy for this black hole,  $\lambda$  is required to be the same with  $\delta$ . By recovering the dimension, the minimal length is obtained as  $2\sqrt{\lambda} = 2\sqrt{\delta} \ell_p \approx 0.127484 \times \ell_p$ , where  $\ell_p$  is the Plank length.

In conclusion, our calculation shows that it is possible to obtain the positive definite convergent entropy in the near horizon limit of the RN black hole by using the brick wall method along with the generalized uncertainty principle. Conversely, the requirement of the area law determines the arbitrary positive constant  $\lambda$  in the generalized uncertainty principle. And, the most interesting thing to distinguish from the conventional method is that there is no divergence at all, which seems to be a generic behavior as long as we use the generalized uncertainty relation.

## Acknowledgments

This work was supported by the Science Research Center Program of the Korea Science and Engineering Foundation through the Center for Quantum Spacetime (CQUeST) of

- [1] J. D. Bekenstein, Lett. Nuovo. Cim. **4**, 737 (1972); Phys. Rev. D **7**, 2333 (1973); Phys. Rev. D **9**, 3292 (1974).
- [2] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
- [3] W. H. Zurek and K. S. Thorne, Phys. Rev. Lett. **54**, 2171 (1985).
- [4] V. P. Frolov and I. Novikov, Phys. Rev. D **48**, 4545 (1993); V. P. Frolov and D. V. Fursaev, Phys. Rev. D **61**, 024007 (1999).
- [5] C. G. Callan and F. Wilczek, Phys. Lett. B **333**, 55 (1994).
- [6] R. B. Mann, *Quantum Entropy of Charged Rotating Black Hole* [gr-qc/9607024].
- [7] G. 't Hooft, Nucl. Phys. B **256**, 727 (1985).
- [8] L. Susskind and J. Uglum, Phys. Rev. D **50**, 2700 (1994); T. Jacobson, Phys. Rev. D **50**, R6031 (1994).
- [9] J-G. Demers, R. Lafrance, and R. C. Myers, Phys. Rev. D **52**, 2245 (1995).
- [10] M-H. Lee, H-C. Kim, and J. K. Kim, Phys. Lett. B **388**, 487 (1996); S-W. Kim, W. T. Kim, Y-J. Park, and H. Shin, Phys. Lett. B **392**, 311 (1997); J. Ho, W. T. Kim, Y-J. Park, and H. Shin, Class. Quant. Grav. **14**, 2617 (1997); J. Ho and G. Kang, Phys. Lett. B **445**, 27 (1998)
- [11] S. Mukohyama, Phys. Rev. D **61**, 124021 (2000).
- [12] X. Li, Phys. Lett. B **540**, 9 (2002).
- [13] W-B. Liu, Chin. Phys. Lett. **20**, 440 (2003); C-Z. Liu, Int. J. Theor. Phys. **44**, 567 (2005).
- [14] W. Liu, Y. Han, and Z. Zhou, Int. J. Mod. Phys. A **18**, 2681 (2003).
- [15] X-F. Sun and W-B. Liu, Mod. Phys. Lett. A **19**, 677 (2004).
- [16] W. Kim, Y-W. Kim, and Y-J. Park, Phys. Rev. D **74**, 104001 (2006); J. Kor. Phys. Soc. **49**, 1360 (2006).
- [17] L. J. Garay, Int. J. Mod. Phys. A **10**, 145 (1995).
- [18] A. Kempf, G. Mangano, and R. B. Mann, Phys. Rev. D **52**, 1108 (1995); L. N. Chang, D. Minic, N. Okamura, and T. Takeuchi, Phys. Rev. D **65**, 125028 (2002).